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GENERALIZATIONS OF 1-ABSORBING PRIMARY IDEALS OF COMMUTATIVE RINGS

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Let R be a commutative ring with identity. In this paper, we extend the concept of 1-absorbing primary ideals to the concept of φ-1-absorbing primary ideals. Let $\varphi: S(R) \to S(R) \cup \{\emptyset\}$ be a function, where $S(R)$ is the set of all ideals of R. A proper ideal I of R is called a φ-1-absorbing primary ideal of R if whenever nonunit elements a, b, c of R and $abc \in I \backslash \varphi(I)$, then $ab \in I$ or $c \in \sqrt{I}$. A number of results concerning φ-1-absorbing primary ideals are given and some characterizations of φ-1-absorbing primary ideals in some special rings are obtained.

Keywords: 1-absorbing primary ideal, weakly 1-absorbing primary ideal, φ-1-absorbing primary ideal.

1. Introduction

Throughout this paper $\boldsymbol{R}$ denotes a commutative ring with identity and the set of all ideals of $\boldsymbol{R}$ is denoted by $\boldsymbol{S(R)}$.Various generalizations of prime (primary) ideals are studied in [1]-[14]. The concept of weakly prime ideals was introduced by Anderson and Smith in [2]. According to that paper, a proper ideal $\boldsymbol{I}$ of $\boldsymbol{R}$ is called a weakly prime ideal of $\boldsymbol{R}$ if whenever $\boldsymbol{0 \neq ab \in I}$ for some $\boldsymbol{a, b \in I}$, then $\boldsymbol{a \in I}$ or $\boldsymbol{b \in I}$. A proper ideal $\boldsymbol{I}$ of $\boldsymbol{R}$ is called a weakly primary ideal of $\boldsymbol{R}$ as in [4] if whenever $\boldsymbol{0 \neq ab \in I}$ for some $\boldsymbol{a, b \in I}$, then $\boldsymbol{a \in I}$ or $\boldsymbol{b \in \sqrt{I}}$. Let $\boldsymbol{\varphi: S(R) \to S(R) \cup \{\emptyset\}}$ be a function. A proper ideal $\boldsymbol{I}$ of $\boldsymbol{R}$ is called a $\boldsymbol{\varphi}$-prime ideal of $\boldsymbol{R}$ as in [1] if whenever $\boldsymbol{a, b \in R}$ with $\boldsymbol{ab \in I \setminus \varphi(I)}$ implies $\boldsymbol{a \in I}$ or $\boldsymbol{b \in I}$. A proper ideal $\boldsymbol{I}$ of $\boldsymbol{R}$ is called a $\boldsymbol{\varphi}$-primary ideal of $\boldsymbol{R}$ as in [12] if whenever $\boldsymbol{a, b \in R}$ with $\boldsymbol{ab \in I \setminus \varphi(I)}$, then $\boldsymbol{a \in I}$ or $\boldsymbol{b \in \sqrt{I}}$. As a different generalization of (resp. weakly prime) prime ideals, (resp. [6]) [5] that a proper ideal $\boldsymbol{I}$ of $\boldsymbol{R}$ is said to be a (resp. weakly) 2-absorbing ideal of $\boldsymbol{R}$ if whenever (resp. $\boldsymbol{0 \neq abc \in I}$) $\boldsymbol{abc \in I}$ for some $\boldsymbol{a, b, c \in R}$, then $\boldsymbol{ab \in I}$ or $\boldsymbol{bc \in I}$ or $\boldsymbol{ac \in I}$. Recall that a proper ideal $\boldsymbol{I}$ of $\boldsymbol{R}$ is called a $\boldsymbol{\varphi}$-2-absorbing ideal of $\boldsymbol{R}$ as in [13] if whenever $\boldsymbol{a, b, c \in R}$ with $\boldsymbol{abc \in I \setminus \varphi(I)}$, then $\boldsymbol{ab \in I}$ or $\boldsymbol{ac \in I}$ or $\boldsymbol{bc \in I}$. Some generalizations of (resp. weakly) 2-absorbing ideals, (resp. weakly) 2-absorbing primary ideals were introduced and studied in [7] and [8], where a proper ideal of $\boldsymbol{R}$ is said to be a (resp. weakly) 2-absorbing primary ideal if whenever $\boldsymbol{a, b, c \in R}$ with (resp. $\boldsymbol{0 \neq abc \in I}$) $\boldsymbol{abc \in I}$,

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then $\boldsymbol{ab} \in \boldsymbol{I}$ or $\boldsymbol{ac} \in \sqrt{\boldsymbol{I}}$ or $\boldsymbol{bc} \in \sqrt{\boldsymbol{I}}$. Later, a generalization of 2-absorbing primary ideals was introduced which covers all the above mentioned definitions. Recall from [9] that a proper ideal $\boldsymbol{I}$ of $\boldsymbol{R}$ is said to be a $\boldsymbol{\varphi}$-2-absorbing primary ideal if whenever $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \boldsymbol{R}$ with $\boldsymbol{abc} \in \boldsymbol{I} \setminus \boldsymbol{\varphi}(\boldsymbol{I})$, then $\boldsymbol{ab} \in \boldsymbol{I}$ or $\boldsymbol{ac} \in \sqrt{\boldsymbol{I}}$ or $\boldsymbol{bc} \in \sqrt{\boldsymbol{I}}$. In some recent studies [10] and [11], Badawi and Yetkin Celikel introduced the concepts of 1-absorbing primary and weakly 1-absorbing primary ideals. A proper ideal $\boldsymbol{I}$ of $\boldsymbol{R}$ is said to be a (resp. weakly) 1-absorbing primary if whenever nonunit $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \boldsymbol{R}$ and (resp. $\mathbf{0} \neq \boldsymbol{abc} \in \boldsymbol{I}$) $\boldsymbol{abc} \in \boldsymbol{I}$, then $\boldsymbol{ab} \in \boldsymbol{I}$ or $\boldsymbol{c} \in \sqrt{\boldsymbol{I}}$.

In this paper, we extend the concept of 1-absorbing primary ideal to the concept of $\boldsymbol{\varphi}$-1-absorbing primary ideal. Let $\boldsymbol{\varphi}: \boldsymbol{S}(\boldsymbol{R}) \to \boldsymbol{S}(\boldsymbol{R}) \cup \{\emptyset\}$ be a function. We call a proper ideal $\boldsymbol{I}$ of $\boldsymbol{R}$ a $\boldsymbol{\varphi}$-1-absorbing primary ideal if whenever nonunit elements $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ of $\boldsymbol{R}$ with $\boldsymbol{abc} \in \boldsymbol{I} \setminus \boldsymbol{\varphi}(\boldsymbol{I})$, then $\boldsymbol{ab} \in \boldsymbol{I}$ or $\boldsymbol{c} \in \sqrt{\boldsymbol{I}}$. Let $\boldsymbol{\varphi}(\boldsymbol{I}) \neq \emptyset$. Then, we have the following relations among these mentioned concepts:

$$\begin{array}{ccccc}
\text{primary} & \Rightarrow & \text{1-absorbing primary} & \Rightarrow & \text{2-absorbing primary} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{weakly primary} & \Rightarrow & \text{weakly 1-absorbing primary} & \Rightarrow & \text{weakly 2-absorbing primary} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\boldsymbol{\varphi}\text{-primary} & \Rightarrow & \boldsymbol{\varphi}\text{-1-absorbing primary} & \Rightarrow & \boldsymbol{\varphi}\text{-2-absorbing primary}
\end{array}$$

We show that $\boldsymbol{\varphi}$-1-absorbing primary ideals enjoy analogs of many of the properties of (weakly) 1-absorbing primary ideals. Among many results in this paper, we show (Theorem 2.3) that if $\boldsymbol{I}$ is a $\boldsymbol{\varphi}$-1-absorbing primary ideal of $\boldsymbol{R}$ such that $\boldsymbol{\varphi}$ is an order-preserving function with $\boldsymbol{\varphi}(\boldsymbol{I}) \neq \boldsymbol{I}$ and $\sqrt{\boldsymbol{\varphi}(\boldsymbol{I})} = \boldsymbol{\varphi}(\boldsymbol{I})$, then $\sqrt{\boldsymbol{I}}$ is a prime ideal of $\boldsymbol{R}$. Additionally, if $\boldsymbol{R}$ is a Dedekind domain, we show (Theorem 2.5) that the converse of Theorem 2.3 is also satisfied. We show (Theorem 2.4) that in divided rings (chained rings) $\boldsymbol{\varphi}$-1-absorbing primary ideals $\boldsymbol{I}$ of $\boldsymbol{R}$ coincide with $\boldsymbol{\varphi}$-primary ideals of $\boldsymbol{R}$ provided that $\boldsymbol{\varphi}(\boldsymbol{I}) \neq \boldsymbol{I}$ and $\boldsymbol{\varphi}(\boldsymbol{I})$ is a radical ideal, where $\boldsymbol{\varphi}$ is an order-preserving function. Furthermore, a characterization for $\boldsymbol{\varphi}$-1-absorbing primary ideals in u-rings is obtained via equivalent conditions. (Theorem 2.9).

2. φ-1-absorbing primary ideals

Definition 2.1. Let I be a φ-1-absorbing primary ideal of R. We define the function $\varphi_\alpha: S(R) \to S(R) \cup \{\emptyset\}$ and the corresponding φ_α-1-absorbing primary ideals, where $\alpha \in \{\emptyset\} \cup \{0,1,2,\dots\} \cup \{\omega\}$ as follows:
(1) If $\varphi(I) = \emptyset$ for every $I \in S(R)$, then $\varphi = \varphi_\emptyset$ and I is called a $\varphi_\emptyset$-1-absorbing primary ideal of R, and hence I is a 1-absorbing primary ideal of R.
(2) If $\varphi(I) = 0$ for every $I \in S(R)$, then $\varphi = \varphi_0$ and I is called a φ_0-1-absorbing primary ideal of R. Thus I is a weakly 1-absorbing primary ideal of R.

(3) If $\varphi(I) = I$ for every $I \in S(R)$, then $\varphi = \varphi_1$ and I is called a φ_1-1-absorbing primary ideal of R. (any ideal.)
(4) If $n \geq 2$ and $\varphi(I) = I^{\mathrm{n}}$ for every $I \in S(R)$, then $\varphi = \varphi_n$ and I is called a φ_n-1-absorbing primary ideal (n-almost 1-absorbing primary ideal) of R. In particular, if $n = 2$, then we say that I is an almost-1-absorbing primary ideal
(5) If $\varphi(I) = \bigcap_{n=1}^{\infty} I^n$ for every $I \in S(R)$, then $\varphi = \varphi_\omega$ and I is called a φ_ω-1-absorbing primary ideal of R.

Since $I \setminus \varphi(I) = I \setminus (I \cap \varphi(I))$, throughout we may assume that $\varphi(I) \subseteq I$. Given two functions $\psi_1, \psi_2 : S(R) \to S(R) \cup \{\emptyset\}$, if $\psi_1(I) \subseteq \psi_2(I)$, then it is clear that if I is a ψ_1-1-absorbing primary, then I is a ψ_2-1-absorbing primary ideal of R. The following example shows that the concepts of (weakly) 1-absorbing primary ideal and φ-1-absorbing primary ideal are different.

Example 2.1. Consider an ideal $J = \{0,6,12,18\}$ of $\mathbb{Z}_{24}$ and the idealization ring $R = \mathbb{Z}_{24}(+)J$. Then $I = \{(0,0), (0,6), (0,12), (0,18)\}$ is a φ_2 -1-absorbing primary ideal of R. Observe that if $abc \in I$ for some $a, b, c \in R\backslash I$, abc can be neither (0,6) nor (0,18). So, $abc \in I$ for some $a, b, c \in R\backslash I$ if and only if $abc \in \varphi_2(I) = I^2 = \{(0,0), (0,12)\}$. Thus I is a φ_2-1-absorbing primary ideal of R. However, it is not weakly 1-absorbing primary since $(0,0) \neq (0,12) = (2,0)(4,6)(3,0) \in I$, but neither $(2,0)(4,6) = (8,12) \in I$ nor $(3,0) \in \sqrt{I} = I$.

Theorem 2.1. Let I be a proper ideal of R. Then
(1) I is a primary ideal $\Rightarrow$ I is a 1-absorbing primary ideal $\Rightarrow$ I is a weakly 1-absorbing primary ideal $\Rightarrow$ I is a φ_ω-1- absorbing primary ideal $\Rightarrow$ I is an n-almost-1-absorbing primary ideal for $n \geq 2$ $\Rightarrow$ I is an almost 1-absorbing primary ideal.
(2) If I is an idempotent ideal of R, then I is an φ_ω-1-absorbing primary ideal of R.
(3) I is a φ_n-1-absorbing primary ideal of R for all $n \geq 2$ if and only if I is a φ_ω-1-absorbing primary ideal of R.
(4) If $R/\varphi(I)$ is an integral domain, then I is a φ-1-absorbing primary ideal of R if and only if I is a 1-absorbing primary ideal of R.
(5) If R is a quasilocal ring with maximal ideal $\sqrt{0}$, then I is a φ-1-absorbing primary ideal of R for all φ.

Proof. (1) It is clear that every primary ideal is 1-absorbing primary. Since $\emptyset \subseteq 0 \subseteq \bigcap_{n=1}^{\infty} I^n \subseteq \cdots \subseteq I^{\mathrm{n}} \subseteq \cdots \subseteq I^2 \subseteq I$, the implications follow from the definitions.
(2) If I is an idempotent ideal, then $I = I^2 = I^{\mathrm{n}} = \bigcap_{n=1}^{\infty} I^n$; so we are done.
(3) It is obvious from definitions.
(4) Suppose that I is a φ-1-absorbing primary ideal of R. Let $abc \in I$ for some nonunit $a, b, c \in R$ and $ab \notin I$. If $abc \notin \varphi(I)$, then $c \in \sqrt{I}$, so we are done. Assume that $abc \in \varphi(I)$. Since $ab \notin \varphi(I)$ and $R/\varphi(I)$ is an integral domain, we concude that $c \in \varphi(I) \subseteq I \subseteq \sqrt{I}$. The converse part is clear.
(5) If R is a quasilocal ring with maximal ideal $\sqrt{0}$, then for every nonunit element c of R, $c \in \sqrt{0} \subseteq \sqrt{I}$. Therefore, I is a φ-1-absorbing primary ideal of R.

Theorem 2.2. Let $\varphi: S(R) \to S(R) \cup \{\emptyset\}$ be a function and I a proper ideal of R. Then the following statements hold:
(1) If I is a φ-1-absorbing primary ideal of R and J is a proper ideal of R with $\varphi(I) \subseteq J \subseteq I$, then I/J is a weakly 1-absorbing primary ideal of R/J.
(2) Suppose that the set of unit elements of $R/\varphi(I)$ is $U(R/\varphi(I)) = \{a + \varphi(I) \mid a \in U(R)\}$. Then I is a φ-1-absorbing primary ideal of R if and only if $I/\varphi(I)$ is a weakly 1-absorbing primary ideal of $R/\varphi(I)$.
(3) Suppose that the set of unit elements of R/I^{n} is $U(R/I^{\mathrm{n}}) = \{a + I^{\mathrm{n}} \mid a \in U(R)\}$. Then I is a φ_n-1-absorbing primary ideal of R for $n \geq 2$ if and only if I/I^{n} is a weakly 1-absorbing primary ideal of R/I^{n}.
Proof. (1) Suppose that $J \neq (a+J)(b+J)(c+J) \in I/J$ for some nonunit $(a+J)$, $(b+J), (c+J) \in R/J$. Hence $abc \in I \setminus J$, where a, b, c are nonunit elements of R. Since $\varphi(I) \subseteq J$, we have $abc \in I \setminus \varphi(I)$. It follows that $ab \in I$ or $c \in \sqrt{I}$. Since $\sqrt{I/J} = \sqrt{I}/J$, we have $ab + J \in I/J$ or $c + J \in \sqrt{I}/J$.
(2) Suppose that $I/\varphi(I)$ is a weakly 1-absorbing primary ideal of $R/\varphi(I)$ and $abc \in I \setminus \varphi(I)$ for some nonunit $a, b, c \in R$. Thus $\varphi(I) \neq (a + \varphi(I))(b + \varphi(I))(c + \varphi(I)) \in I/\varphi(I)$, where $a + \varphi(I)$, $b + \varphi(I)$, $c + \varphi(I)$ are nonunit elements of $R/\varphi(I)$ by hypothesis. Hence $(a + \varphi(I))(b + \varphi(I)) \in I/\varphi(I)$ or $c + \varphi(I) \in \sqrt{I}/\varphi(I)$. Thus $ab \in I$ or $c \in \sqrt{I}$. The converse part is clear by (1).
(3) Since $\varphi_n(I) = I^{\mathrm{n}}$, the conclusion follows from (2).

A function $\varphi: S(R) \to S(R) \cup \{\emptyset\}$ is said to be order-preserving if whenever $I, J \in S(R)$ with $I \subseteq J$, then $\varphi(I) \subseteq \varphi(J)$.

Theorem 2.3. Let $\varphi: S(R) \to S(R) \cup \{\emptyset\}$ be an order-preserving function. If I is a φ-1-absorbing primary ideal of R such that $\varphi(I) \neq I$ and $\sqrt{\varphi(I)} = \varphi(I)$, then $\sqrt{I}$ is a prime ideal of R.
Proof. First we show that $\sqrt{I}$ is a φ-prime ideal of R. Suppose that $ab \in \sqrt{I} \setminus \varphi(\sqrt{I})$ for some $a, b \in R$. We may assume that a, b are nonunit. Then there exists an even positive integer $n = 2m$ $(m \geq 1)$ such that $(ab)^{\mathrm{n}} \in I$. If $(ab)^{\mathrm{n}} \in \varphi(I)$, then $ab \in \sqrt{\varphi(I)} = \varphi(I) \subseteq \varphi(\sqrt{I})$, a contradiction. So, we have $(ab)^{\mathrm{n}} = a^m a^m b^{\mathrm{n}} \in I \setminus \varphi(I)$. Thus $a^m a^m = a^{\mathrm{n}} \in I$ or $b^{\mathrm{n}} \in \sqrt{I}$, and therefore $\sqrt{I}$ is a φ-prime ideal of R. Thus, $\sqrt{I}$ is a prime ideal of R by [1, Corollary 7].

Recall that a ring R is called divided if for every prime ideal P of R and for every $x \in R \setminus P$, we have $x \mid p$ for every $p \in P$. We have the following result.

Theorem 2.4. Let R be a divided ring and I a proper ideal of R with such that $\varphi(I) \neq I$ and $\sqrt{\varphi(I)} = \varphi(I)$ for some order-preserving function φ. Then I is a φ-1-absorbing primary ideal of R if and only if I is a φ-primary ideal of R.
Proof. Suppose that $ab \in I \setminus \varphi(I)$ for some $a, b \in R$ and $b \notin \sqrt{I}$. We may assume that a and b are nonunit. Since $\sqrt{I}$ is a prime ideal of R by Theorem 2.3, we

conclude that $a \in \sqrt{I}$. Since R is divided, we conclude that $b \mid a$. Thus $a = bc$ for some $c \in R$. Observe that c is a nonunit element of R as $b \notin \sqrt{I}$ and $a \in \sqrt{I}$. Since $ab = bcb \in I \setminus \varphi(I)$, I is φ-1-absorbing primary and $b \notin \sqrt{I}$, we conclude that $bc = a \in I$. Thus I is a φ-primary ideal of R. The converse part is clear.

Remark 2.1. Recall that a ring R is called a chained ring if for every $x, y \in R$, we have $x \mid y$ or $y \mid x$. Every chained ring is divided. So, if R is a chained ring, then a proper ideal I of R with the properties that $\varphi(I) \neq I$ and $\sqrt{\varphi(I)} = \varphi(I)$, where φ is a order-preserving function, is a φ-1-absorbing primary if and only if it is φ-primary.

Theorem 2.5. Let R be a Dedekind domain and I a proper ideal of R such that $\varphi(I) \neq I$ and $\sqrt{\varphi(I)} = \varphi(I)$, where φ is an order-preserving function. Then I is a φ-1-absorbing primary ideal of R if and only if $\sqrt{I}$ is a prime ideal of R.

Proof. Suppose that I is a φ-1-absorbing primary ideal of R. Then $\sqrt{I}$ is a prime ideal of R by Theorem 2.3. Conversely, if $\sqrt{I}$ is prime, then I is 1-absorbing primary by [10, Theorem 14]. Thus I is a φ-1-absorbing primary ideal of R.

The following result is similar to [11, Theorem 2].

Theorem 2.6. Let I be a φ-1-absorbing primary ideal of R. If I is not a φ-primary ideal of R, then there exist an irreducible element $x \in R$ and a nonunit element $y \in R$ such that $xy \in I \setminus \varphi(I)$, but neither $x \in I$ nor $y \in \sqrt{I}$. Furthermore, if $ab \in I \setminus \varphi(I)$ for some nonunit $a, b \in R$ such that neither $a \in I$ nor $b \in \sqrt{I}$, then a is an irreducible element of R.

Proof. Suppose that I is not a φ-primary ideal of R. Then there exist nonunit $x, y \in R$ such that $xy \in I \setminus \varphi(I)$ with $x \notin I$, $y \notin \sqrt{I}$. Assume that x is not an irreducible element of R. Then $x = cd$ for some nonunit $c, d \in R$. Since $xy = cdy \in I \setminus \varphi(I)$ and I is φ-1-absorbing primary and $y \notin \sqrt{I}$, we conclude that $cd = x \in I$, a contradiction. Hence x is an irreducible element of R.

Theorem 2.7. Let x be a nonunit element of R with $(0:x) \subseteq (x)$. Then the following statements are equivalent:

(1) (x) is φ-1-absorbing primary ideal of R for some φ with $\varphi \leq \varphi_3$.

(2) (x) is a 1-absorbing primary ideal of R.

Proof. (1)⇒(2) Suppose that (x) is φ_3-1-absorbing primary and $abc \in (x)$ and $ab \notin (x)$ for some nonunit $a, b, c \in R$. If $abc \notin (x^3) = \varphi_3((x))$, we have $c \in \sqrt{(x)}$. Suppose that $abc \in (x^3)$. Hence $abc + abx = ab(c + x) \in (x)$. Since $ab \notin (x)$, $(c + x)$ is a nonunit element of R. Now, we show that $ab(c + x) \notin (x^3)$. Assume that $ab(c + x) \in (x^3)$. Since $abc \in (x^3)$, we have $abx \in (x^3)$. Hence there exists $r \in R$ such that $abx = rx^3$ which means $(ab - rx^2) \in (0:x)$. We get $ab \in (0:x) + (x) \subseteq (x)$, a contradiction. Thus $ab(c + x) \notin (x^3)$. Since (x) is φ_3-

1-absorbing primary and $ab \notin (x)$, we have $(c+x) \in \sqrt{(x)}$. Thus $c \in \sqrt{(x)}$. Therefore, (x) is a 1-absorbing primary ideal of R. (2)⇒(1) is clear.

Theorem 2.8. Let I be a φ-1-absorbing primary ideal of R with $(\varphi(I):c) \subseteq \varphi((I:c))$. If c is a nonunit element of $R\backslash I$, then $(I:c)$ is a φ-primary ideal of R.

Proof. Suppose that $ab \in (I:c) \setminus \varphi((I:c))$ and $a \notin (I:c)$. Hence b is a nonunit element of R. If a is unit, then $b \in (I:c) \subseteq \sqrt{(I:c)}$ and we are done. Assume that a is nonunit. Then $acb \in I \setminus \varphi(I)$ as $(\varphi(I):c) \subseteq \varphi((I:c))$. Since $ac \notin I$ and I is a φ-1-absorbing primary ideal of R, we conclude that $b \in \sqrt{I} \subseteq \sqrt{(I:c)}$, as needed.

Let R be a commutative ring. If an ideal of R contained in a finite union of ideals must be contained in one of those ideals, then R is called a u-ring [15]. The next theorem gives a characterization for φ-1-absorbing primary ideals in u-rings.

Theorem 2.9. Let I be a proper ideal of a u-ring R. The following are equivalent:

(1) I is a φ-1-absorbing primary ideal of R.

(2) For every nonunit $a, b \in R$ with $ab \notin I$, $(I:ab) = (\varphi(I):ab)$ or $(I:ab) \subseteq \sqrt{I}$.

(3) For every nonunit $a \in R$ and every ideal I_1 of R with $I_1 \nsubseteq \sqrt{I}$, if $(I:aI_1)$ is a proper ideal of R, then $(I:aI_1) = (\varphi(I):aI_1)$ or $(I:aI_1) \subseteq (I:a)$.

(4) For every ideals I_1, I_2 of R with $I_1 \nsubseteq \sqrt{I}$, if $(I:I_1I_2)$ is a proper ideal of R, then $(I:I_1I_2) = (\varphi(I):I_1I_2)$ or $(I:I_1I_2) \subseteq (I:I_2)$.

(5) For every ideals I_1, I_2, I_3 of R with $I_1I_2I_3 \subseteq I \setminus \varphi(I)$, $I_1I_2 \subseteq I$ or $I_3 \subseteq \sqrt{I}$.

Proof. (1)⇒(2) Suppose that $c \in (I:ab)$. Then $abc \in I$. Since $ab \notin I$, c is nonunit. If $abc \in \varphi(I)$, then $c \in (\varphi(I):ab)$. Assume that $abc \in I \setminus \varphi(I)$. Since I is φ-1-absorbing primary, we have $c \in \sqrt{I}$. Hence $(I:ab) \subseteq (\varphi(I):ab) \cup \sqrt{I}$. Since R is a u-ring, we obtain that $(I:ab) = (\varphi(I):ab)$ or $(I:ab) \subseteq \sqrt{I}$. (2)⇒(3) Suppose that $c \in (I:aI_1)$. It is clear that c is nonunit. Then $acI_1 \subseteq I$. Now $I_1 \subseteq (I:ac)$. If $ac \in I$, then $c \in (I:a)$. Suppose that $ac \notin I$. Hence $(I:ac) = (\varphi(I):ac)$ or $(I:ac) \subseteq \sqrt{I}$ by (2). Thus $I_1 \subseteq (\varphi(I):ac)$ or $I_1 \subseteq \sqrt{I}$. Since $I_1 \nsubseteq \sqrt{I}$ by hypothesis, we conclude $I_1 \subseteq (\varphi(I):ac)$; i.e. $c \in (\varphi(I):aI_1)$. Thus $(I:aI_1) \subseteq (\varphi(I):aI_1) \cup (I:a)$. Since R is a u-ring, we have $(I:aI_1) = (\varphi(I):aI_1)$ or $(I:aI_1) \subseteq (I:a)$. (3)⇒(4) If $I_1 \subseteq \sqrt{I}$, then we are done. Suppose that $I_1 \nsubseteq \sqrt{I}$ and $c \in (I:I_1I_2)$. Then $I_2 \subseteq (I:cI_1)$. Since $(I:I_1I_2)$ is proper, c is nonunit. Hence $I_2 \subseteq (\varphi(I):cI_1)$ or $I_2 \subseteq (I:c)$ by (3). If $I_2 \subseteq (\varphi(I):cI_1)$, then $c \in (I:I_1I_2)$. If $I_2 \subseteq (I:c)$, then $c \in (I:I_2)$. So, $(I:I_1I_2) \subseteq (\varphi(I):I_1I_2) \cup (I:I_2)$ which implies that $(I:I_1I_2) = (\varphi(I):I_1I_2)$ or $(I:I_1I_2) \subseteq (I:I_2)$. (4)⇒(5) It is clear. (5)⇒(1) Let $a, b, c \in R$ nonunit elements and $abc \in I \setminus \varphi(I)$. Put $I_1 = aR$, $I_2 = bR$, and $I^3 = cR$ in (5). Then the result is clear.

Definition 2.2. Let I be a φ-1-absorbing primary ideal of R. If $abc \in \varphi(I)$ for some nonunit $a, b, c \in R$ such that $ab \notin I$, $c \notin \sqrt{I}$, then we call (a, b, c) a φ-1-triple-zero of I.

Let I be a proper ideal of a ring R. Observe that I is φ-1-absorbing primary that is not 1-absorbing primary if and only if I has a φ-triple-zero. We obtain the next theorem and corollary similar to [11,Theorem 10 and Theorem 11].

Theorem 2.10. Let I be a φ-1-absorbing primary ideal of R, and (a, b, c) a φ-1-triple-zero of I. Then

(1) $abI \subseteq \varphi(I)$.

(2) If $a, b \notin (I: c)$, then $bcI = acI = aI^2 = bI^2 = cI^2 \subseteq \varphi(I)$.

(3) If $a, b \notin (I: c)$, then $I^3 \subseteq \varphi(I)$.

Proof. (1) Suppose that $abI \nsubseteq \varphi(I)$. Then $abx \notin \varphi(I)$ for some nonunit $x \in I$. Hence $ab(c + x) \in I \nsubseteq \varphi(I)$. Since $ab \notin I, (c + x)$ is nonunit element of R. Since I is a φ-1-absorbing primary ideal of R and $ab \notin I$, we conclude that $(c + x) \in \sqrt{I}$. Since $x \in I$, we have $c \in \sqrt{I}$, a contradiction. Thus $abI \subseteq \varphi(I)$.

(2) Suppose that $bcy \notin \varphi(I)$ for some nonunit $y \in I$. Hence $bcy = b(a + y)c \in I \setminus \varphi(I)$. Since $b \notin (I: c)$, we conclude that $a + y$ is a nonunit element of R. Since I is φ-1-absorbing primary, $ab \notin I$ and $by \in I$, we conclude that $b(a + y) \notin I$, and hence $c \in \sqrt{I}$, a contradiction. Thus $bcI \subseteq \varphi(I)$. We show that $acI \subseteq \varphi(I)$. Suppose that $acI \nsubseteq \varphi(I)$. Then $acy \notin \varphi(I)$ for some nonunit element $y \in I$. Hence $acy = a(b + y)c \in I \setminus \varphi(I)$. Since $a \notin (I: c)$, we conclude that $b + y$ is a nonunit element of R. Since I is φ-1-absorbing primary, $ab \notin I$ and $ay \in I$, we conclude that $a(b + y) \notin I$, and hence $c \in \sqrt{I}$, a contradiction. Thus $acI \subseteq \varphi(I)$. Now we prove that $aI^2 \subseteq \varphi(I)$. Suppose that $axy \notin \varphi(I)$ for some $x, y \in I$. Since $abI \subseteq \varphi(I)$ by (1) and $acI \subseteq \varphi(I)$ by (2), $axy = a(b + x)(c + y) \in I \setminus \varphi(I)$. Since $ab \notin I$, we conclude that $c + y$ is a nonunit element of R. Since $a \notin (I: c)$, we conclude that $b + x$ is a nonunit element of R. Since I is φ-1-absorbing primary, we have $a(b + x) \in I$ or $(c + y) \in \sqrt{I}$. Since $x, y \in I$, we conclude that $ab \in I$ or $c \in \sqrt{I}$, a contradiction. Thus $aI^2 \subseteq \varphi(I)$. By a similar manner, one can easily obtain that $bI^2 \subseteq \varphi(I)$. Now, we show that $cI^2 \subseteq \varphi(I)$. Suppose that $cxy \notin \varphi(I)$ for some $x, y \in I$. Since $acI = bcI \subseteq \varphi(I)$ by (2), $cxy = (a + x)(b + y)c \in I \setminus \varphi(I)$. Since $a, b \notin (I: c)$, we conclude that $a + x$ and $b + y$ are nonunit elements of R. Since I is φ-1-absorbing primary, we have $(a + x)(b + y) \in I$ or $c \in \sqrt{I}$. Since $x, y \in I$, we conclude that $ab \in I$ or $c \in \sqrt{I}$, a contradiction. Thus $cI^2 \subseteq \varphi(I)$.

(3) Assume that $xyz \nsubseteq \varphi(I)$ for some $x, y, z \in I$. Then $xyz = (a + x)(b + y)(c + z) \in I \setminus \varphi(I)$ by (1) and (2). Since $ab \notin I$, we conclude $c + z$ is a nonunit element of R. Since $a, b \notin (I: c)$, we conclude that $a + x$ and $b + y$ are nonunit elements of R. Since I is φ-1-absorbing primary, $(a + x)(b + y) \in I$ or $c + z \in \sqrt{I}$. Since $x, y, z \in I$, we conclude that $ab \in I$ or $c \in \sqrt{I}$, a contradiction. Thus $I^3 \subseteq \varphi(I)$.

Corollary 2.1. Let I be a φ-1-absorbing primary ideal of R such that $\varphi(I) \neq I$.

(1) Let $R/\varphi(I)$ be a reduced ring. If (a,b,c) is a φ-1-triple-zero of I, then $ac \in I$ or $bc \in I$.
(2) If $(a+\varphi(I), b+\varphi(I), c+\varphi(I))$ is a 1-triple-zero of $I/\varphi(I)$, then (a,b,c) is a φ-1-triple-zero of I.
Proof. (1) Assume that neither $ac \in I$ nor $bc \in I$. Hence $I^3 \subseteq \varphi(I)$ by Theorem 2.10 (3). Since $R/\varphi(I)$ is reduced, we conclude that $I \subseteq \varphi(I)$. Thus $I = \varphi(I)$, a contradiction. Therefore, $ac \in I$ or $bc \in I$.
(2) Suppose that $((a+\varphi(I), b+\varphi(I), c+\varphi(I))$ is a 1-triple-zero of $I/\varphi(I)$. It is clear that a,b,c are nonunit elements of R. Then $((a+\varphi(I))(b+\varphi(I)) \notin I/\varphi(I)$ and $c+\varphi(I) \notin \sqrt{I/\varphi(I)} = \sqrt{I}/\varphi(I)$. Thus $ab \notin I$ and $c \notin \sqrt{I}$ which means that (a,b,c) is a φ-1-triple-zero of I.

The following theorem is a generalization of [11, Theorem 15].

Theorem 2.11. Let R and R' be commutative rings with identity, $f: R \to R'$ be a ring homomorphismwith $f(1_R) = 1_{R'}$, and let $\varphi: S(R) \to S(R) \cup \{\emptyset\}$ and $\varphi': S(R') \to S(R') \cup \{\emptyset\}$ be functions. Then the following statements hold:
(1) Suppose that $f(a)$ is a nonunit element of R' for every nonunit $a \in R$ (for example if $U(R')$ is a torsion group) and J is a φ'-1-absorbing primary ideal of R' with $\varphi(f^{-1}(J)) = f^{-1}(\varphi'(J))$. Then $f^{-1}(J)$ is a φ-1-absorbing primary ideal of R.
(2) If f is an epimorphism, I is a φ-1-absorbing primary ideal of R such that $Ker(f) \subseteq I$ and $\varphi'(f(I)) = f(\varphi(I))$, then $f(I)$ is a φ-1-absorbing primary ideal of R'.
Proof. (1) Let $abc \in f^{-1}(J) \setminus \varphi(f^{-1}(J))$ for some nonunit $a,b,c \in R$. Then $f(abc) = f(a)f(b)f(c) \in J \setminus \varphi'(J)$, where $f(a), f(b), f(c)$ are nonunit elements of R' by hypothesis. Hence $f(a)f(b) \in J$ or $f(c) \in \sqrt{J}$. Hence $ab \in f^{-1}(J)$ or $c \in \sqrt{f^{-1}(J)} = f^{-1}(\sqrt{J})$. Thus $f^{-1}(J)$ is a φ-1-absorbing primary ideal of R.
(2) Let $xyz \in f(I) \setminus \varphi'(f(I))$ for some nonunit $x,y,z \in R$. Since f is onto, there exists nonunit $a,b,c \in R$ such that $x = f(a)$, $y = f(b)$, $z = f(c)$. Then $f(abc) = f(a)f(b)f(c) = xyz \in f(I) \setminus f(\varphi(I))$. Since $Ker(f) \subseteq I$, we have $abc \in I \setminus \varphi(I)$. It follows $ab \in I$ or $c \in \sqrt{I}$. Thus $xy \in f(I)$ or $z \in f(\sqrt{I})$. Since f is onto and $Ker(f) \subseteq I$, we have $f(\sqrt{I}) = \sqrt{f(I)}$. Thus we are done.

Given an ideal I of R, define $\varphi_I: S(R/I) \to S(R/I) \cup \{\emptyset\}$ by $\varphi_I(J/I) = (\varphi(J)+I)/I$ for $I \subseteq J$. Note that $\varphi_I(J/I) \subseteq J/I$ and $(\varphi_\alpha)_I = \varphi_\alpha$ for $\alpha \in \{\emptyset\} \cup \{0,1,2,\dots\}$.

Corollary 2.2. Let I and J be proper ideals of R such that $I \subseteq J$. If J is a φ-1-absorbing primary ideal of R, then J/I is a φ_I-1-absorbing primary ideal of R/I.
Proof. Suppose that $(a+I)(b+I)(c+I) \in J/I \setminus \varphi_I(J/I) = J/I \setminus (\varphi(J)+I)/I$ for some nonunit $a+I, b+I, c+I \in R/I$. Since $I \subseteq J$, we have $abc \in J \setminus \varphi(J)$, where a,b,c are nonunit elements of R. Hence $ab \in J$ or $c \in \sqrt{J}$. Thus $(a+I)(b+I) \in J/I$ or $(c+I) \in \sqrt{J}/I = \sqrt{J/I}$.

We characterize φ-1-absorbing primary ideals of $R = R_1 \times R_2$, where R_1 and R_2 are commutative rings with identity in the following theorem.

Theorem 2.12. Let R_1 and R_2 be commutative rings with identity, I_1, I_2 nonzero ideals of R_1, R_2, respectively, and $R = R_1 \times R_2$. Let $\psi_i: S(R_i) \to S(R_i) \cup \emptyset$ $(i = 1,2)$ be functions such that $\psi_1(I_1) = I_1$ and $\psi_2(I_2) = I_2$. Let $\varphi = \psi_1 \times \psi_2$. If I is a proper ideal of R, then the following statements are equivalent:

(1) $I_1 \times I_2$ is a φ-1-absorbing primary ideal of R.

(2) $I_1 = R_1$, I_2 is a primary ideal of R_2 or $I_2 = R_2$, I_1 is a primary ideal of R_1.

(3) $I_1 \times I_2$ is a primary ideal of R.

(4) $I_1 \times I_2$ is a 1-absorbing primary ideal of R.

Proof. Suppose that $\psi_1(I_1) = \emptyset$ or $\psi_2(I_2) = \emptyset$. Then $\varphi(I_1 \times I_2) = \emptyset$. Hence (1)⇔(2)⇔(3). Thus assume that $\varphi(I_1 \times I_2) = \emptyset$, and hence neither $\psi_1(I_1) = \emptyset$ nor $\psi_2(I_2) = \emptyset$. (1)⇒(2). Suppose that $I_1 \times I_2$ is a φ-1-absorbing primary ideal of R. Hence $I_1/\psi_1(I_1) \times I_2/\psi_2(I_2)$ is a weakly 1-absorbing primary ideal of $R_1/\psi_1(I_1) \times R_2/\psi_2(I_2)$ by Theorem 2.2 (1). Hence by [11, Theorem 13], we conclude that $I_1/\psi_1(I_1) = R_1/\psi_1(I_1)$ and $I_2/\psi_2(I_2)$ is a primary ideal of $R_2/\psi_2(I_2)$ or $I_2/\psi_2(I_2) = R_2/\psi_2(I_2)$ and $I_1/\psi(I_1)$ is a primary ideal of $R_1/\psi_1(I_1)$. Thus $I_1 = R_1$ and I_2 is a primary ideal of R_2 or $I_2 = R_2$ and I_1 is a primary ideal of R_1. (2)⇒(3)⇒(4)⇒(1) It is straightforward.

Definition 2.3. Let $\varphi: S(R) \to S(R) \cup \{\emptyset\}$ be a function, and S a multiplicatively closed subset of R. A proper ideal L_S of R_S, where L is a proper ideal of R such that $L \cap S = \emptyset$, is called a φ_S-1-absorbing primary ideal of R_S if whenever nonunit $a, b, c \in R_S$ with $abc \in L_S \setminus \varphi(L)_S$, then $ab \in L_S$ or $c \in \sqrt{L_S}$.

Let $Z(R)$ denotes the set of all zero-divisors of R and $Z_I(R) = \{a \in R: ab \in I \text{ for some } b \in R\backslash I\}$. We have the following result similar to [11, Theorem 17].

Theorem 2.13. Let I be a proper ideal of R and S a multiplicatively closed subset of R such that $S \cap Z(R) = S \cap I = S \cap Z_I(R) = \emptyset$. Then I is a φ-1-absorbing primary ideal of R if and only if I_S is a φ_S-1-absorbing primary ideal of R_S.

Proof. Suppose that $abc \in I_S \setminus \varphi(I)_S$ for some nonunit $a, b, c \in R_S$. Hence there is an $s \in S$ and nonunit $a_1, b_1, c_1, d_1 \in R$ such that $a = a_1/s, b = b_1/s, c = c_1/s$. Thus $((a_1b_1c_1)/(s^3)) = ((d_1)/(s^3)) \in I_S \setminus \varphi(I)_S$. Since $Z(R) \cap S = \emptyset$, we have $a_1b_1c_1 = d_1 \in I \setminus \varphi(I)$. Thus $ab \in I$ or $c \in \sqrt{I}$. Since $Z(R) \cap S = \emptyset$, $\sqrt{I_S} = (\sqrt{I})_S$. Thus $ab \in I_S$ or $c \in \sqrt{I_S}$. Conversely, suppose that $abc \in I \setminus \varphi(I)$ for some nonunit $a, b, c \in R$. Then $((abc)/1) = (a/1)(b/1)(c/1) \in I_S \setminus \varphi(I)_S$ as $S \cap Z(R) = \emptyset$. Hence $(a/1)(b/1) \in I_S$ or $(c/1) \in \sqrt{I_S}$. If $(a/1)(b/1) \in S^{-1}I$, then $uab \in I$ for some $u \in S$. Since $S \cap Z_I(R) = \emptyset$, we conclude that $ab \in I$. If $(c/1) \in (\sqrt{I})_S$, then $(tc)^{\mathrm{n}} \in I$ for some $n \geq 1$ and $t \in S$. Since $S \cap Z_I(R) = \emptyset$, we have $t^{\mathrm{n}} \notin Z_I(R)$. Hence $c^{\mathrm{n}} \in I$. Thus I is a φ-1-absorbing primary ideal of R.

Definition 2.4. Let I be a φ-1-absorbing primary ideal of R and $I_1 I_2 I_3 \subseteq I$ for some proper ideals I_1, I_2, I_3 of R. If (a, b, c) is not φ-1-triple zero of I for every $a \in I_1, b \in I_2, c \in I_3$, then we call I a free φ-1-triple zero with respect to $I_1 I_2 I_3$.

Theorem 2.14. Let I be a φ-1-absorbing primary ideal of R and J a proper ideal of R with $abJ \subseteq I$ for some $a, b \in R$. If (a, b, j) is not a φ-1-triple zero of I for all $j \in J$ and $ab \notin I$, then $J \subseteq \sqrt{I}$.

Proof. The proof is similar to the proof of [11, Theorem 18].

Corollary 2.3. Let I be a φ-1-absorbing primary ideal of R and $I_1 I_2 I_3 \subseteq I \setminus \varphi(I)$ for some proper ideals $I_1, I_2, I_3\ of\ R$. If I is free φ-1-triple zero with respect to $I_1 I_2 I_3$, then $I_1 I_2 \subseteq I$ or $I_3 \subseteq \sqrt{I}$.

Proof. The proof is similar to the proof of [11, Theorem 19].

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R E F E R E N C E S

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